XVII. *On the Method of correſpondent Values, &c. By* Edward Waring, *M. D. F. R. S. and* Lucaſian *Profeſſor of the Mathematics at* Cambridge.

Read May 28, 1789.

I.

1. IN the year 1762 I publiſhed a method of finding when two roots of a given equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ are equal, by finding the common diviſors of the two quantities $a^n - pa^{n-1} + qa^{n-2} - \&c.$, and $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c.$, and obſerved if they admitted only one ſimple diviſor $(a - A)$, then two roots were only equal; if a quadratic $(a^2 - Aa + B)$, then two roots of the equation became twice equal; if a cubic $(a^3 - Aa^2 + Ba - C)$, then two roots became thrice equal; and ſo on: or, to expreſs in more general terms what follows from the ſame principles, if the common diviſor be $\overline{a-b}^r \times \overline{a-c}^s \times \overline{a-d}^t \times \&c.$, then $r+1$ roots of the given equation will be b, $s+1$ roots will be c, $t+1$ will be d, &c.; and it immediately follows, from the principles delivered in the ſecond edition of the ſame Book, publiſhed in 1770, that to find when $r+1$, $v+1$, $t+1$, &c. roots are reſpectively equal requires $r+s+t$, &c. equations of condition, which are deducible from the well known method of finding the common diviſors of two quantities in this caſe of $a^n - pa^{n-1} + qa^{n-2} - \&c.$, $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c.$ of the terms of their remainders, &c.

In the book above mentioned the equations of condition are given, which diſcover when two roots are equal in the equations $x^3 - px^2 + qx - r = 0$, $x^4 + qx^2 - rx + s = 0$, $x^5 + qx^3 - rx^2 + sx - t = 0$, in the two latter equations the ſecond term is wanting, which may eaſily be exterminated; but it may as eaſily be reſtored by ſubſtituting for q, r, s, &c. in the equation of condition found the quantities reſulting from the common transformation of equations to deſtroy the ſecond term.

2. Another rule contained in the ſame Book is the ſubſtitution of the roots of the equation $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c. = 0$ reſpectively for a in the quantity $a^n - pa^{n-1} + qa^{n-2} - \&c.$, and multiplication of all the quantities reſulting into each other; their content will give the equation of condition, when two roots are equal.

Mr. HUDDE firſt diſcovered, that if the ſucceſſive terms of the given equation are multiplied into an arithmetical ſeries, the reſulting equation will contain one of any two equal roots, and m of the $\overline{m+1}$ equal roots in the given equation.

3. If 3, 4, 5, . . r roots of the equation are equal, find a common diviſor of 3, 4, 5, . . r of the ſubſequent quantities $a^n - pa^{n-1} + qa^{n-2} - \&c.$, $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c.$, $n \,.\, \overline{n-1}a^{n-2} - \overline{n-1} \,.\, \overline{n-2}pa^{n-3} + \overline{n-2} \,.\, \overline{n-3}qa^{n-4} - \overline{n-3} \,.\, \overline{n-4}ra^{n-5} + \&c.$, $n \,.\, \overline{n-1} \,.\, \overline{n-2}a^{n-3} - \overline{n-1} \,.\, \overline{n-2} \,.\, \overline{n-3}pa^{n-4} + \overline{n-2} \,.\, \overline{n-3} \,.\, \overline{n-4}qa^{n-5} - \&c.$, . . . $n \,.\, \overline{n-1} \,.\, \overline{n-2} \,.\,.\, \overline{n-r+2}a^{n-r+1} - \overline{n-1} \,.\, \overline{n-2} \,.\,.\, \overline{n-r+1}pa^{n-r} + \&c.$; which will probably be beſt done by dividing all the preceding quantities by the quantity of the leaſt dimenſion of a, and the diviſor and all the remainders by that quantity which has the leaſt dimenſions amongſt them; and ſo on: there will reſult 2, 3, 4, . . $r - 1$ equations of condition; and in this caſe it is

obſerved,

obſerved, in the before-mentioned Book, that (if the common diviſor be $(a - A)$) it will once only admit of $3, 4, 5, \ldots r$ equal roots; if it be a quadratic, then it will twice admit of thoſe equal roots; and ſo on.

4. If the roots of the equation of the leaſt dimenſions be ſubſtituted for a in the remaining equations, and each of the reſulting values of the ſame equation be multiplied into each other, there will reſult the $r - 1$ equations of condition: and the ſame may be deduced alſo from the ſeveral equations conjointly.

The equations of conditions found by the firſt method, if the diviſions were not properly inſtituted, may admit of more rational diviſors than neceſſary, of which ſome are the equations of conditions required.

2.

1. In the year 1776, I publiſhed in the Meditationes Analyticæ a new method of differences for the reſolution of the following problem.

Given the ſums of a ſwiftly converging ſeries $ax + bx^2 + cx^3 + dx^4 + \&c.$, when the values of x are reſpectively π, ρ, ς, &c.; to find the ſum of the ſeries when x is τ, that is, given $S\pi = a\pi + b\pi^2 + c\pi^3 + d\pi^4 + \&c$, $S\varrho = a\varrho + b\rho^2 + c\rho^3 + \&c.$, $S\sigma = a\sigma + b\sigma^2 + c\sigma^3 + \&c.$ &c.; to find $S\tau = a\tau + b\tau^2 + c\tau^3 + \&c.$

To reſolve this problem I multiplied the quantities, $S\pi$, $S\rho$, $S\sigma$, &c. reſpectively into unknown co-efficients α, β, γ, &c. and there reſulted

$$\begin{array}{l}\alpha\pi a+\alpha\pi^2 b+\alpha\pi^3 c+\&c.\\ \beta\rho a+\beta\rho^2 b+\beta\rho^3 c+\&c.\\ \gamma\sigma a+\gamma\sigma^2 b+\gamma\sigma^3 c+\&c.\\ \quad\&c. \quad \&c. \quad \&c.\end{array}$$

and then made the ſum of each of the terms reſpectively equal to its correſpondent term of the quantity $\tau a+\tau^2 b+\tau^3 c+\&c.$, and conſequently $\alpha\pi+\beta\rho+\gamma\sigma+\&c.=\tau$, $\alpha\pi^2+\beta\rho^2+\gamma\sigma^2+\&c.=\tau^2$, $\alpha\pi^3+\beta\varrho^3+\gamma\sigma^3+\&c.=\tau^3$, &c. I aſſumed as many equations of this kind as there were given values π, ϱ, σ, &c. of x; and conſequently as many equations reſulted as unknown quantities α, β, γ, &c.; whence, by the common reſolution of ſimple equations, or more eaſily from differences, can be found the unknown quantities α, β, γ, &c., and thence the equation ſought $\alpha\times S\pi+\beta\times S\varrho+\gamma\times S\sigma+\&c.=S\tau$ nearly.

3. In the Meditationes are aſſumed for π, ϱ, σ, &c. the quantities p, $2p$, $3p$, $4p$, $\ldots\overline{n-2}p$, $\overline{n-1}p$, and np for τ; which, if ſubſtituted for their values in the preceding equations, will give $\alpha+2\beta+3\gamma+4\delta+\&c.=n$, $\alpha+4\beta+9\gamma+16\delta+\&c.=n^2$, $\alpha+8\beta+27\gamma+\&c.=n^3$, $\alpha+16\beta+81\gamma+\&c.=n^4$; and if the ſums of the ſeries $ax+bx^2+cx^3+\&c.$ which reſpectively correſpond to the values p, $2p$, $3p$, $\ldots\overline{n-1}p$ of x be $S1$, $S2$, $S3$, $S4$, $\ldots S\overline{n-1}$, and the ſum of the ſeries $ax+bx^2+cx^3+\&c.$ which correſponds to n value of x be Sn; then will $Sn=nS\overline{n-1}-n.\frac{n-1}{2}S\overline{n-2}+n.\frac{n-1}{2}.\frac{n-2}{2}S\overline{n-3}\ldots\pm nS1$ nearly, which equation is given in the above-mentioned Book.

3. The logarithm from the number, the arc from the ſine, &c. are found by ſerieſes of the formula $ax+bx^2+cx^3+\&c.$; and conſequently this equation is applicable to them.

4. In the ſame Book is aſſumed a ſeries $ax^{r}+bx^{r+s}+cx^{r+2s}+dx^{r+3s}+$ &c. of a more general formula than the preceding, and in it for x ſubſtituted α, β, γ, δ, &c., m; and $S\alpha$, $S\beta$, $S\gamma$, $S\delta$, &c; Sm for the reſulting ſums, and thence deduced

$$Sm = \frac{m^{r}\times\overline{m^{s}-\beta^{s}}\,.\,\overline{m^{s}-\gamma^{s}}\,.\,\overline{m^{s}-\delta^{s}}\,.\,\&c.}{\alpha^{r}\times\overline{\alpha^{s}-\beta^{s}}\,.\,\overline{\alpha^{s}-\gamma^{s}}\,.\,\overline{\alpha^{s}-\delta^{s}}\,.\,\&c.}\times S\alpha + \frac{m^{r}\times\overline{m^{s}-\alpha^{s}}\,.\,\overline{m^{s}-\gamma^{s}}\,.\,\overline{m^{s}-\delta^{s}}\,.\,\&c.}{\beta^{r}\times\overline{\beta^{s}-\alpha^{s}}\,.\,\overline{\beta^{s}-\gamma^{s}}\,.\,\overline{\beta^{s}-\delta^{s}}\,.\,\&c.}$$

$$\times S\beta + \frac{m^{r}\times\overline{m^{s}-\alpha^{s}}\,.\,\overline{m^{s}-\beta^{s}}\,.\,\overline{m^{s}-\delta^{s}}\,.\,\&c.}{\gamma^{r}\times\overline{\gamma^{s}-\alpha^{s}}\,.\,\overline{\gamma^{s}-\beta^{s}}\,.\,\overline{\gamma^{s}-\delta^{s}}\,.\,\&c.}\times S\gamma + \frac{m^{r}\times\overline{m^{s}-\alpha^{s}}\,.\,\overline{m^{s}-\beta^{s}}\,.\,\overline{m^{s}-\gamma^{s}}\times\&c.}{\delta^{r}\times\overline{\delta^{s}-\alpha^{s}}\,.\,\overline{\delta^{s}-\beta^{s}}\,.\,\overline{\delta^{s}-\gamma^{s}}\,.\,\&c.}$$

$\times S\delta +$ &c. nearly.

Cor. If for r and s be aſſumed reſpectively 1, the ſeries becomes $ax+bx^{2}+cx^{3}+$ &c. of the ſame formula as the preceding: if $r=0$ and $s=1$, the ſeries becomes $a+bx+cx^{2}+$ &c. The latter caſe will be the ſame as the former, when one of the quantities (α) ſubſtituted for x and its correſpondent ſum $S\alpha$, both become $=0$, and the equation deduced in both caſes the ſame.

5. If π, ρ, σ, &c. reſpectively denote r, $r+p$, $r+2p$, ... $r+\overline{n-2}p$, $r+\overline{n-1}p$, and $\tau=r+np$; and S, S1, S2, S3, ... $S\overline{n-2}$, $S\overline{n-1}$, be the ſums either reſulting from the ſeries $ax+bx^{2}+cx^{3}+$ &c. or the ſeries $A+ax+bx^{2}+cx^{3}+$ &c., which reſpectively correſpond to the values r, $r+p$, $r+2p$, &c. of x; and Sn the ſum of the ſame ſeries which correſponds to the value $r+np$ of x; then will $Sn = nS\overline{n-1} - n\,.\,\frac{n-1}{2}S\overline{n-2} + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}S\overline{n-3} - \ldots \pm n\,.\,\frac{n-1}{2}S2 \mp nS1 \pm S$ nearly; this equation differs from the preceding by the laſt term S not vaniſhing; in the preceding caſe S became $=0$, for it was the ſum of the ſeries $ax+bx^{2}+cx^{3}+$ &c., which correſponded to $x=0$.

6. From the Meditationes it appears, that $r^m - n \times \overline{r \pm p}^m + n \cdot \frac{n-1}{2} \overline{r \pm 2p}^m - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \overline{r \pm 3p}^m +$ &c. to the end of the ſeries $= 0$, if m is leſs than n, and m and n are whole numbers; but if $m = n$, then it will $= \pm 1 \cdot 2 \cdot 3 \cdot 4 \ldots \overline{n-1} \cdot np^m$; whence it is manifeſt, that for the n firſt terms of the ſeries $A + ax + bx^2 +$ &c. the equations are true; and for the $n-1$ firſt terms of the ſeries $ax + bx^2 + cx^3 +$ &c. and in the ſucceſſive term of both the ſerieſes they will err by a quantity nearly $= \pm 1 \cdot 2 \cdot 3 \ldots n \times p^n \times r^{-n} \times$ co-efficient of the term; and the errors of every ſubſequent term (x^{4+n}) will be nearly as $\pm m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \ldots \frac{m-h+1}{h} \times p^n \times r^{-n} \times$ co-efficient of the term x^{h+n}, if for r, $r+p$, $r+2p$, &c. be ſubſtituted 1, $1 + \frac{p}{r}$, $1 + \frac{2p}{r}$, &c.

7. Let the preceding equation $Sn = nS\overline{n-1} - n \cdot \frac{n-1}{2} S\overline{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot S\overline{n-3} -$ &c. $= n \times \log. \overline{r-p} - n \cdot \frac{n-1}{2} \log. \overline{r-2p} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \log. \overline{r-3p} +$ &c. $= \log. \frac{r \times \overline{r-2p}^{s} \times \overline{r-4p}^{s'} \times \overline{r-6p}^{s''} \times \&c.}{\overline{r-p}^{t} - \overline{r-3p}^{t'} \times \overline{r-5p}^{t''} \times \&c.} = \log.\ K$, where s, s', s'', &c. denote the co-efficients of the alternate terms of the binomial theorem, *viz.* $s = n \cdot \frac{n-1}{2}$, $s' = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$, &c., and $t = n$, $t' = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, &c. the co-efficients of the remaining alternate terms; the numerator $r \times \overline{r-2p}^{s} \times \overline{r-4p}^{s'} \times \overline{r-6p}^{s''} \times$ &c. $=$ (if $N = 2^{n-1}$) $r^N - Ppr^{N-1} + Qp^2r^{N-2} - Rp^3r^{N-3} \ldots Lp^{n-1} \times r^{N-n+1} \pm Mp^n r^{N-n} \mp$ &c.; and the denominator $\overline{r-p}^{t} \times \overline{r-3p}^{t'} \times \overline{r-5p}^{t''} \times$ &c. $= r^N - Ppr^{N-1} + Qp^2r^{N-2} -$

$Rp^3r^{n-3} + \dots Lp^{n-1}r^{N-n+1}(\pm M + 1 . 2 . 3 \dots \overline{n-1})\, p^n r^{N-n} \mp$ &c., whence the numerator and denominator have the n firſt terms the ſame, and the next ſucceeding terms differ by $1 . 2 . 3 \dots \overline{n-1}\, p^n r^{N-n}$; the numerator divided by the denominator $= 1 \pm \frac{1 . 2 . 3 \dots \overline{n-1}}{r^n} p^n$ nearly, if r be a great number in proportion to p, &c. it would be + when n is an odd number, and − when even.

8. The logarithm of the fraction K by the common ſeries $= K - 1 - \frac{\overline{K-1}^2}{2} + \frac{\overline{K-1}^3}{3} -$ &c. has for its firſt term $= \pm \frac{1 . 2 . 3 \dots \overline{n-1}}{r^n} \times p^n$ nearly; for its ſecond term the ſquare of the firſt divided by 2, &c.

9. The error of this equation not only depends on the logarithm of K, which may be calculated to any degree of exactneſs, but in the calculus on the errors of the given logarithms.

10. If r be increaſed or diminiſhed by any given number, the n firſt terms of the numerator and denominator will ſtill reſult the ſame, and the next ſucceeding terms will differ by $1 . 2 . 3 . 4 \dots n-1 \times p^n \times r^{N-n}$.

11. Let $n . \frac{n-1}{2}$ numbers be 2, $n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4}$ numbers be 4, $n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} . \frac{n-4}{5} . \frac{n-5}{6}$ numbers be 6, &c.; their ſum, the ſum of the products of every two, the contents of every three, four, five, &c. to $n-1$ of them will be equal to the ſum, the ſum of the products of every two, of the contents of every three, four, five, &c. to $\overline{n-1}$ of the following numbers, *viz.* n numbers which are 1, $n . \frac{n-1}{2} . \frac{n-2}{3}$

numbers which are 3, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5}$, which are 5, &c.; and the ſum of the contents of every n of the former will be leſs than the ſum of the contents of every n latter numbers by $1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \overline{n-1}$.

12. The method given in Art. 4. which I name a method of correſpondent values, eaſily deduces and demonſtrates the preceding equations, which cannot, without much difficulty, be done by the preceding method of differences; the method of correſpondent values is much preferable to the method of differences, both for the facility of its deduction, and the generality of its reſolution: for inſtance, from this method very eaſily can be deduced, &c. the ſubſequent and other ſimilar equations.

Ex. 1. $Sn = nS\overline{n-1} - n \cdot \frac{n-1}{2} S\overline{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} S\overline{n-3} \cdot \cdot \cdot \pm nS1 \mp S$ nearly.

Ex. 2. $S\overline{n+m} = \frac{\overline{m+n} \cdot \overline{m+n-1} \cdot \overline{m+n-2} \cdot \cdot \cdot \overline{m+2}}{1 \cdot 3 \cdot \cdot \overline{n-1}} \times S\overline{n-1} - \frac{n-1}{1} \times A \times \frac{m+1}{m+2} \times S\overline{n-2} + \frac{n-2}{2} \times B \times \frac{m+2}{m+3} S\overline{n-3} - \frac{n-3}{3} \times C \times \frac{m+3}{m+4} \times S\overline{n-4} + \frac{n-4}{4} \times D \times \frac{m+4}{m+5} \times S\overline{n-5} -$ &c. nearly, where the letters A, B, C, D, &c. denote the preceding co-efficients, and the converging ſeries is the ſame as in the preceding example.

Ex. 3. Let the converging ſeries be of the formula $ax + bx^3$ $cx^5 + dx^7 +$ &c.; then will $Sn = \overline{2n-2}\, S\overline{n-1} - \overline{2n-1} \times \frac{2n-4}{2} S\overline{n-2} + \overline{2n-1} \times \frac{2n-2}{2} \times \frac{2n-6}{3} S\overline{n-3} - \overline{2n-1} \cdot \frac{2n-2}{2} \cdot \frac{2n-3}{2} \times \frac{2n-8}{4} S\overline{n-4} +$ &c. nearly, of which the general term is $\overline{2n-1} \cdot \frac{2n-2}{2} \cdot \frac{2n-3}{3} \cdot \cdot \frac{2n-l+1}{l-1} \times \frac{2n-2l}{l} \times S\overline{n-l}$.

Ex. 4. Let the ſeries be of the formula $A + ax^2 + bx^4 + cx^6 +$ &c.; then will $Sn = \frac{n-1}{n} \times \overline{2n-2}\, S\overline{n-1} - \frac{n-2}{n} \times \overline{2n-1} . \frac{2n-4}{2} \times S\overline{n-2} + \frac{n-3}{n} \times \overline{2n-1} . \frac{2n-2}{2} . \frac{2n-6}{3} S\overline{n-3} - \frac{n-4}{n} \times \overline{2n-1} . \frac{2n-2}{2} . \frac{2n-3}{3} \times \frac{2n-8}{4} S\overline{n-4} +$ &c. nearly, of which the general term is $\frac{n-l}{n} \times \overline{2n-1} . \frac{2n-2}{3} . \frac{2n-3}{3} \ldots \frac{2n-l+1}{l-1} \times \frac{2n-2l}{l} \times S\overline{n-l}$.

Ex. 5. Let the given ſeries be of the formula $ax + bx^2 + cx^3 +$ &c., and in it for x be ſubſtituted p, $-p$, $2p$, $-2p$, $3p$, $-3p$, …, np, $-np$ and mp, and for the ſums of the reſulting ſerieſes be wrote reſpectively S^1, S^{-1}, S^2, S^{-2}, S^3, S^{-3}, … S^n, S^{-n}, and Sm; then will $Sm =$

$$\frac{m . \overline{m^2-1} . \overline{m^2-4} . \overline{m^2-9} . \overline{m^2-16} \ldots \overline{m^2-\overline{n-1}^2} . \overline{m-n}}{n . \overline{n^2-1} . \overline{n^2-4} . \overline{n^2-9} . \overline{n^2-16} \ldots \overline{n^2-\overline{n-1}^2} \times 2n = 1 . 2 . 3 . 4 . . 2n}$$

$\times S^{-n} + A \times \frac{m+n}{m-n} S^{+n} - \frac{2n}{1} \times B \times \frac{m-n}{m+n-1} S^{-n+1} - C \times \frac{m+n-1}{m-\overline{n-1}} \times S^{n-1} + \frac{2n-1}{2} \times D \times \frac{m-\overline{n-1}}{m+n-2} \times S^{-n+2} + \frac{m+n-2}{m-\overline{n-2}} \times E\, S^{n-2} - \frac{2n-2}{3} \times F \times \frac{m-\overline{n-2}}{m+\overline{n-3}} S^{-n+3} - G \times \frac{m+n-3}{m-\overline{n-3}} \times S^{n-3} +$ &c. nearly, where the letters A, B, C, D, &c. reſpectively denote the preceding co-efficients. In general, the co-efficients of the terms S^{-n+s} and S^{n-s} will be reſpectively $M = \frac{2n-s+1}{s} \times L \times \frac{m-\overline{n-s+1}}{m+n-s}$ and $M \times \frac{m+n-s}{m-\overline{n-s}}$, where the letters L and M reſpectively denote their preceding co-efficients; the co-efficients are to be taken affirmatively, or negatively, according as s is an even or odd number.

Ex. 6. If for x in the preceding ſeries be ſubſtituted p, $-p$, $2p$, $-2p$, $3p$, $-3p$, … $\overline{n-1}p$, $-\overline{n-1}p$, np reſpectively, then

$\mathrm{S}m = \frac{m\,.\,\overline{m^2-1}\,.\,\overline{m^2-4}\,.\,\overline{m^2-9}\,.\,\overline{m^2-16}\,\ldots\,(m^2-(n-1)^2)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,\ldots\,(2n-1)}\,\mathrm{S}^n - \mathrm{A} \times \frac{m-n}{m+n-1} \times \mathrm{S}^{-n+1} - \frac{2n-1}{1} \times \mathrm{B} \times \frac{m+n-1}{m-n-1} \times \mathrm{S}^{n-1} + \mathrm{C} \times \frac{m-\overline{n-1}}{m+n-2} \times \mathrm{S}^{-n+2} + \frac{2n-2}{2} \times \mathrm{D} \times \frac{m+n-2}{m-n-2} \times \mathrm{S}^{n-2}$ – &c. nearly, when A, B, C, &c. denote as before the preceding co-efficients. The coefficients of the terms S^{-n+s} and S^{n-s} will be reſpectively $\mathrm{L} \times \frac{m-\overline{n-s+1}}{m+n-s}$ and $\frac{2n-1}{s} \times \mathrm{M} \times \frac{m+\overline{n-s}}{m-n-s}$, L and M denoting the preceding co-efficients, which are to be taken negatively or affirmatively, as s is an even or an odd number. In this ſeries when $x = 0$, the correſpondent ſum $= 0$.

Ex. 7. Let the given ſeries be of the formula $a + bx + cx^2 + dx^3$ + &c.; and in it for x be ſubſtituted 0, p, $-p$, $2p$, $-2p$, $3p$, $-3p$ . . . np, $-pp$ and mp, and for the ſums of the reſulting ſerieſes be wrote as before S^0, S^1, S^{-1}, S^2, S^{-2}, . . . S^n, S^{-n}, and S^m; then will $\mathrm{S}m = \frac{m\,.\,\overline{m^2-1}\,.\,\overline{m^2-4}\,.\,\overline{m^2-9}\,\ldots\,(m^2-\overline{n-1}^2) \times \overline{m-n}}{n\,.\,\overline{n^2-1}\,.\,\overline{n^2-4}\,.\,\overline{n^2-9}\,\ldots\,(n^2-\overline{n-1}^2) \times 2n} \times \mathrm{S}^{-n} + \mathrm{A} \times \frac{m+n}{m-n} \times \mathrm{S}^{+n} - \frac{2n}{1} \times \mathrm{B} \times \frac{m-n}{m+n-1}\,\mathrm{S}^{-n+1}$ – &c. this ſeries obſerves the ſame law as the ſeries given in Ex. 5. and only differs from it by the laſt term $\mathrm{S}0$ not vaniſhing, that is, being $= 0$.

Ex. 8 Let the ſeries be of the preceding formula $a + bx + cx^2 + dx^3$ + &c., and in it for x be ſubſtituted 0; p, $-p$; $2p$, $-2p$; $3p$, $-3p$; . . . $\overline{n-1}p$, $-\overline{n-1}p$, np, and mp, and the ſums reſulting be $\mathrm{S}0$, $\mathrm{S}1$, S^{-1}, S^2, S^{-2}, . . $\mathrm{S}^{\overline{n-1}}$, S^{-n+1}, S^n and S^m; then will $\mathrm{S}m = \frac{m\,.\,\overline{m^2-1}\,.\,\overline{m^2-4}\,.\,.\,\overline{m^2-(n-1)^2}}{1\,.\,2\,.\,3\,.\,4\,\ldots\,\overline{2n-1}}\,\mathrm{S}^n - \mathrm{A} \times \frac{m-n}{m+n-1} \times \mathrm{S}^{-n+1}$ – &c. the ſame ſeries as in Ex. 6. and differs from it only by the laſt term $\mathrm{S}0$ not vaniſhing.

Ex. 9. Let the ſeries be of the ſame formula $a+bx+cx^2+dx^3+$ &c. and in it for x be ſubſtituted p, $-p$, $3p$, $-3p$, $5p$, $-5p$, $7p$, $-7p$, ... np, $-np$ and mp; and the ſums reſulting be S^1, S^{-1}, S^3, S^{-3}, S^5, S^{-5}, S^7, S^{-7}, ... S^n, S^{-n}, and S^m; then will $S^m =$

$$\frac{\overline{m^2-1}\,.\,\overline{m^2-9}\,.\,\overline{m^2-25}\,.\,\overline{m^2-49}\,\ldots\,\overline{m^2-\overline{n-2}^2}\times\overline{m+n}}{\overline{n^2-1}\,.\,\overline{n^2-9}\,.\,\overline{n^2-25}\,.\,\overline{n^2-49}\,\ldots\,\overline{n^2-\overline{n-2}^2}^2\times 2n = 2^n\times 1\,.\,2\,.\,3\,.\,4\,.\,5\,..\,n}$$

$$\times S^n - A\times\frac{m-n}{m+n}\times S^{-n}-\frac{n}{1}\times B\times\frac{m+n}{m-n+2}\times S^{n-2}+C\times\frac{m-n+2}{m+n-2}\times$$

$$S^{-n+2}+\frac{n-1}{2}\times D\times\frac{m+n-2}{m-n+4}\times S^{n-4}-E\times\frac{m-n+4}{m+n-4}S^{-n+4}-\frac{n-2}{3}F$$

$$\times\frac{m+n-4}{m-n+6}S^{n-6}+\frac{m-n+6}{m+n-6}\times G\times S^{-n+6}+\frac{n-3}{4}\times H\times\frac{m+n-6}{m-n+8}\times$$

S^{n-8} − &c. nearly, where the letters A, B, C, D, E, &c. denote the preceding co-efficients of the terms S^n, S^{-n}, S^{n-2}, S^{-n+2}, S^{n-4}, S^{-n+4}, S^{n-6}, &c. reſpectively. The co-efficients of the terms S^{2n-2s} and S^{-n+2s} will be $M=\frac{n-s+1}{s}\times L\times\frac{m+n-2s+2}{m-n+2s}$ and $N=M\times\frac{m-n+2s}{m+n-2s}$; where L, M, and N denote the co-efficients of the terms immediately preceding each other, that is, of the terms $S^{-n+2s-2}$, S^{n-2s}, and S^{-n+2s}. The ſign of the firſt co-efficient M will be + or −, according as s is even or odd; the ſecond term N will have a contrary ſign to the firſt.

Theſe ſeries may be made to begin from any term, which may be eaſily found by the method of correſpondent values, and the ſubſequent terms from it by the given law; its preceding terms may be deduced from the ſame law reverſed, that is, by putting the numerators of the fractions multiplied into it for the denominators, and the denominators for the numerators.

From theſe different ſerieſes may be formed, by adding two or more terms of the given ſeries together for a term of the required ſeries; which method has been applied to converging ſeries in general in the Meditationes.

13. The method of correſpondent values eaſily affords a reſolution of the problems contained in Mr. Brigg's or Sir Isaac Newton's method of differences.

Ex. 1. Let the quantity be of the formula $a+bx+cx^2+dx^3+\&c.\ldots x^n=y$, and $n+1$ correſpondent values of x and y be given, *viz.* p, q, r, s, &c. of x; $\mathrm{S}p$, $\mathrm{S}q$, $\mathrm{S}r$, $\mathrm{S}s$, &c. of y; then will $y=\frac{\overline{x-q}\,.\,\overline{x-r}\,.\,\overline{x-s}\,.\,\&c.}{\overline{p-q}\,.\,\overline{p-r}\,.\,\overline{p-s}\,.\,\&c.}\times\mathrm{S}p+\frac{\overline{x-p}\,.\,\overline{x-r}\,.\,\overline{x-s}\,.\,\&c.}{\overline{q-p}\,.\,\overline{q-r}\,.\,\overline{q-s}\,.\,\&c.}\times\mathrm{S}q+\frac{\overline{x-p}\,.\,\overline{x-q}\,.\,\overline{x-s}\,.\,\&c.}{\overline{r-p}\,.\,\overline{r-q}\,.\,\overline{r-s}\,.\,\&c.}\times\mathrm{S}r+\frac{\overline{x-p}\,.\,\overline{x-q}\,.\,\overline{x-r}\,.\,\&c.}{\overline{s-p}\,.\,\overline{s-q}\,.\,\overline{s-r}\,.\,\&c.}\times\mathrm{S}s+\&c.$

The truth of this problem very eaſily appears by writing p, q, r, s, &c. for x in the given ſeries.

All the preceding examples may be applied to this caſe, by writing x for m in the given ſeries; hence the reſolutions of ſeveral caſes of equi-diſtant ordinates by eaſy and not inelegant ſerieſes, amongſt which are included the two caſes commonly given on this ſubject.

14. If a quantity be required, which proceeds according to the dimenſions of x, reduce the above given value of y into a quantity proceeding according to the dimenſions of x, and there reſults $y=\left(\frac{\mathrm{S}p}{\overline{p-q}\;\overline{p-r}\,.\,\overline{p-s}\,.\,\&c.=\mathrm{A}}+\frac{\mathrm{S}q}{\overline{q-p}\,.\,\overline{q-r}\,.\,\overline{q-s}\,.\,\&c.=\mathrm{B}}+\frac{\mathrm{S}r}{\overline{r-p}\,.\,\overline{r-q}\,.\,\overline{r-s}\,.\,\&c.=\mathrm{C}}+\frac{\mathrm{S}s}{\overline{s-p}\,.\,\overline{s-q}\,.\,\overline{s-r}\,.\,\&c.=\mathrm{D}}+\&c.\right)\times x^n-\left(\frac{\mathrm{S}p\times\overline{q+r+s+\&c.}}{\mathrm{A}}+\frac{\mathrm{S}q\times\overline{p+r+s+\&c.}}{\mathrm{B}}+\frac{\mathrm{S}r\times\overline{p+q+s+\&c.}}{\mathrm{C}}+\frac{\mathrm{S}s\times\overline{p+q+r+\&c.}}{\mathrm{D}}+\&c.\right)x^{n-1}+\left(\frac{\mathrm{S}p\times\overline{qr+qs+rs+\&c.}}{\mathrm{A}}+\right.$

Sq

$$\frac{\mathrm{S}q\times\overline{pr+ps+rs+\&c.}}{\mathrm{B}}+\frac{\mathrm{S}r\times\overline{pq+ps+qs+\&c.}}{\mathrm{C}}+\frac{\mathrm{S}s\times\overline{pq+pr+qr+\&c.}}{\mathrm{D}}+\&c.)$$

$$\times\, x^{n-2}-\left(\frac{\mathrm{S}p\times\overline{qrs+\&c.}}{\mathrm{A}}+\frac{\mathrm{S}q\times\overline{prs+\&c.}}{\mathrm{B}}+\frac{\mathrm{S}r\times\overline{pqs+\&c.}}{\mathrm{C}}+\frac{\mathrm{S}s\times\overline{pqr+\&c.}}{\mathrm{D}}+\right.$$

$\left.\&c.\right)x^{n-3}+\&c.$

The law and continuation of this ſeries is evident to any one verſant in theſe matters from inſpection.

Theſe fractions may be reduced to a common denominator by ſubſtituting for Sp and A the products Sp × P and A × P, where P $=\overline{q-r}\,.\,\overline{q-s}\,.\,\overline{r-s}\,.$ &c.; for Sq and B the products Sq × Q and B × Q, where Q $=\overline{p-r}\,.\,\overline{p-s}\,.\,\overline{r-s}\,.$ &c.; for Sr and C the products Sr × R and C × R, where R $=\overline{p-q}\,.\,\overline{p-s}\,.\,\overline{q-s}\,.$ &c.; for Ss and D the products Ss × S′ and C × S′, where S′ $=\overline{p-q}\,.\,\overline{p-r}\,.\,\overline{q-r}\,.$ &c. &c.

The fractions, in particular caſes, will often be reducible to lower terms.

15. Let $y=ax^{b}+bx^{b+l}+cx^{b+2l}+\&c.$, and the correſpondent values of x and y be given as before, then will $y=$

$$\frac{x^{b}\times\overline{x^{l}-q^{l}}\times\overline{x^{l}-r^{l}}\times\overline{x^{l}-s^{l}}\times\&c.}{p^{b}\times\overline{p^{l}-q^{l}}\times\overline{p^{l}-r^{l}}\times\overline{p^{l}-s^{l}}\times\&c.}\times\mathrm{S}p+\frac{x^{b}\times\overline{x^{l}-p^{l}}\times\overline{x^{l}-r^{l}}\times\overline{x^{l}-s^{l}}\times\&c.}{q^{b}\times\overline{q^{l}-p^{l}}\times\overline{q^{l}-r^{l}}\times\overline{q^{l}-s^{l}}\times\&c.}\times\mathrm{S}q$$

$$+\frac{x^{b}\times\overline{x^{l}-p^{l}}\times\overline{x^{l}-q^{l}}\times\overline{x^{l}-s^{l}}\times\&c.}{r^{b}\times\overline{r^{l}-p^{l}}\times\overline{r^{l}-q^{l}}\times\overline{r^{l}-s^{l}}\times\&c.}\times\mathrm{S}r+\frac{x^{b}\times\overline{x^{l}-p^{l}}\times\overline{x^{l}-q^{l}}\times\overline{x^{l}-r^{l}}\times\&c.}{s^{b}\times\overline{s^{l}-p^{l}}\times\overline{s^{l}-q^{l}}\times\overline{s^{l}-r^{l}}+\&c.}\times$$

$\mathrm{S}s+\&c.$

This ſeries may in the ſame manner as the preceding be reduced to terms, proceeding according to the dimenſions of x; and the ſerieſes given in the examples may *(mutatis mutandis)* be predicated of it.

16. A more general method of correſpondent values is given in the Meditationes, as alſo the ſubſequent $y=\dfrac{\overline{x-q}\,.\,\overline{x-r}\,.\,\overline{x-s}\,.\,\&c.}{\overline{p-q}\,.\,\overline{p-r}\,.\,\overline{p-s}\,.\,\&c.}$

$\times \mathrm{S}p + \frac{\overline{x-p}\,.\,\overline{x-r}\,.\,\overline{x-s}\,.\,\&c.}{\overline{q-p}\,.\,\overline{q-r}\,.\,\overline{q-s}\,.\,\&c.} \times \mathrm{S}q + \frac{\overline{x-p}\,.\,\overline{x-q}\,.\,\overline{x-s}\,.\,\&c.}{\overline{r-p}\,.\,\overline{r-q}\,.\,\overline{r-s}\,.\,\&c.} \times \mathrm{S}r +$ &c. as in Ex. 1. $= \mathrm{S}p + (x-p)\left(\frac{1}{p-q} \times \mathrm{S}p + \frac{1}{q-p} \times \mathrm{S}q\right) + (x-p)(x-q)\left(\frac{1}{p-q} \times \frac{1}{p-r} \times \mathrm{S}p + \frac{1}{q-p} \times \frac{1}{q-r} \times \mathrm{S}q + \frac{1}{r-p} \times \frac{1}{r-q} \times \mathrm{S}r\right) + (x-p)(x-q)(x-r)\left(\frac{1}{p-q} \cdot \frac{1}{p-r} \cdot \frac{1}{p-s} \cdot \times \mathrm{S}p + \frac{1}{q-p} \cdot \frac{1}{q-r} \cdot \frac{1}{q-s} \times \mathrm{S}q + \frac{1}{r-p} \cdot \frac{1}{r-q} \cdot \frac{1}{r-s} \times \mathrm{S}r + \frac{1}{s-p} \cdot \frac{1}{s-q} \cdot \frac{1}{s-r} \times \mathrm{S}s\right) - \&c.$

The equality of theſe two different quantities will eaſily appear by finding the co-efficients of both, which are multiplied into the ſame given value of y as $\mathrm{S}p$, $\mathrm{S}q$, $\mathrm{S}r$, &c. and the ſame power of x; for with very little difficulty they will in general be found equal.

It is evident from this reſolution that, giving the ordinates and their reſpective diſtances from each other, the value of any other ordinate at a given diſtance from the preceding, found by this method, will reſult the ſame, whatever may be the point aſſumed from which the abſciſs is made to begin.

3.

1. Let a ſeries be $\mathrm{A}x + \mathrm{B}x^2 + \mathrm{C}x^3 + \mathrm{D}x^4 + \&c.$ of ſuch a formula that if in it for x be ſubſtituted $a+b$, there reſults a ſeries $\mathrm{A} \times \overline{a+b} + \mathrm{B} \times \overline{a+b}^2 + \mathrm{C} \times \overline{a+b}^3 + \mathrm{D} \times \overline{a+b}^4 + \&c. = (\mathrm{A}a + \mathrm{B}a^2 + \mathrm{C}a^3 + \mathrm{D}a^4 + \&c.) \times (1 + qb + rb^2 + sb^3 + tb^4 + \&c.) + (1 + qa + ra^2 + sa^3 + ta^4 + \&c.) \times (\mathrm{A}b + \mathrm{B}b^2 + \mathrm{C}b^3 + \mathrm{D}b^4 + \&c.)$ then will the ſeries $\mathrm{A}x + \mathrm{B}x^2 + \mathrm{C}x^3 + \mathrm{D}x^4 + \&c. = \mathrm{A}x + \frac{2\mathrm{B}}{1 \cdot 2}x^2 + \frac{2 \cdot 3\mathrm{C}}{1 \cdot 2 \cdot 3}x^3 + \frac{24\mathrm{ABC} - 8\mathrm{B}^3}{1 \cdot 2 \cdot 3 \cdot 4\mathrm{A}^2}x^4 + \frac{36\mathrm{C}^2\mathrm{A}^2 + 24\mathrm{ACB}^2 - 16\mathrm{B}^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5\mathrm{A}^3}x^5 + \frac{9 \cdot 24\mathrm{A}^2\mathrm{BC}^2 - 4 \times 24\mathrm{AB}^3\mathrm{C}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6\mathrm{A}^4}x^6 + \frac{216\mathrm{C}^3\mathrm{A}^3 + 432\mathrm{A}^2\mathrm{B}^2\mathrm{C}^2 - 384\mathrm{ACB}^4 + 64\mathrm{B}^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7\mathrm{A}^5}x^7 + \&c.;$

$+$&c.; and the ſeries $1+qx+rx^2+sx^3+tx^4+$ &c $= 1+\frac{B}{A}x+$ $\frac{6CA-2B^2}{1.2A^2}x^2+\frac{18CAB-8B^3}{1.2.3A^3}x^3+\frac{36C^2A^2-8B^4}{1.2.3.4A^4}x^4+$ $\frac{180C^2A^2B-120ACB^3+16B^5}{1.2.3.4.5A^5}x^5+\frac{216C^3A^3+216A^2C^2B^2-288ACB^4+64B^6}{1.2.3.4.5.6A^6}x^6$ $+$&c.

The terms of theſe two ſerieſes can eaſily be deduced by the ſubſequent method. Let $Kx^{n-2}+L^{n-1}+Mx^n$, be ſucceſſive terms of the ſeries $Ax+Bx^2+Cx^3+$&c., and $K^1x^{n-2}+L^1x^{n-1}$ ſucceſſive terms of the ſeries $1+qx+rx^2+sx^3+tx^4+$ &c.; then will $M=\frac{2A^2\times B\times K^1+6CAK-2B^2K}{n.\overline{n-1}\times A^2}$ and $L^1=\frac{n\times A\times M-B\times xL}{A^2}$.

Cor. 1. Let $B=0$, and the two ſerieſes $Ax+Bx^2+Cx^3+Dx^4+$ &c. and $1+qx+rx^2+$&c. become reſpectively $Ax+\frac{2.3}{2.3}Cx^3+$ $\frac{2^2.3^2}{2.3.4.5}\times\frac{C^2}{A}x^5+\frac{2^3.3^3}{2.3.4.5.6.7}\times\frac{C^3}{A^2}\times x^7+\frac{2^4.3^4}{2.3.4.5.6.7.8.9}$ $\times\frac{C^4}{A^3}x^9+$ &c., and $1+\frac{2.3}{1.2}\times\frac{C}{A}x^2+\frac{2^2.3^2}{1.2.3.4}\times\frac{C^2}{A^2}x^4+$ $\frac{2^3.3^3}{1.2.3.4.5.6}\times\frac{C^3}{A^3}x^6+$&c.

If in theſe ſerieſes for A be ſubſtituted 1, and for C be ſubſtituted $-\frac{1}{2.3}$, there will reſult the ſerieſes $x-\frac{x^3}{2.3}+$ $\frac{x^5}{2.3.4.5}-$&c., and $1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-$&c. which give the ſine and coſine in terms of the arc x.

Cor. 2. Let $C=0$, and the above-mentioned ſeries $Ax+Bx^2+$ &c. becomes $Ax+\frac{2}{1.2}Bx^2 * -\frac{2^3}{1.2.3.4}\times\frac{B^3}{A^2}x^4-\frac{2^4}{1.2.3.4.5}\times$ $\frac{B^4}{A^3}x^5 * +\frac{2^6}{1.2.3.4.5.6.7}\times\frac{B^6}{A^5}\times x^7+\frac{2^7}{1.2.3.4...8}\times\frac{B^7}{A^6}x^8 * -$ $\frac{2^9}{1.2.3.4..10}\times\frac{B^9}{A^8}x^{10}-\frac{2^{10}}{1.2.3..11}\times\frac{B^{10}}{A^9}x^{11}+$&c. The law of this ſeries is, firſt, that every third term vaniſhes; and, ſecondly, the ſigns of every two ſucceſſive terms change alternately from + to − and − to +; and, laſtly, the co-efficient of the term x^n is $\frac{2^{n-1}}{1 \cdot 2 \cdot 3 \,.\,.\, n} \times \frac{B^{n-1}}{A^{n-2}}$; and the ſeries $1 + qx + rx^2$ + &c. becomes $1 + \frac{B}{A}x - \frac{2B^2}{1 \cdot 2A^2}x^2 - \frac{2^3B^3}{1 \cdot 2 \cdot 3A^3}x^3 - \frac{2^3B^4}{1 \cdot 2 \cdot 3 \cdot 4A^4}x^4 + \frac{2^4B^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5A^5}x^5 + \frac{2^6B^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6A^6}x^6 + \frac{2^6 \times B^7}{1 \cdot 2 \cdot 3 \,.\,.\, 7A^7}x^7 -$ &c. In this ſeries the ſigns of three ſucceſſive terms alternately change from + to − and − to +; and the co-efficient of the term x^n is $\frac{2^n \times B^n}{1 \cdot 2 \cdot 3 \cdot nA^n}$ or $\frac{2^{n-1} \times B^n}{1 \cdot 2 \cdot 3 \,.\,.\, nA^n}$ according as n is diviſible by 3 or not.

2. Let a ſeries $1 + Px + Qx^2 + Rx^3 + Sx^4 + Tx^5 +$ &c. be of ſuch a formula, that if in it for x be ſubſtituted $a+b$, there reſults a ſeries $1 + P \times \overline{a+b} + Q \times \overline{a+b}^2 + R \times \overline{a+b}^3 + S \times \overline{a+b}^4$ + &c. $= (1 + Pa + Qa^2 + Ra^3 + Sa^4 +$ &c.$) \times (1 + Pb + Qb^2 + Rb^3 + Sb^4 +$ &c.$) + (Aa + Ba^2 + Ca^3 + Da^4 +$ &c.$) \times (Ab + Bb^2 + Cb^3 + Db^4 +$ &c.$)$, then will the ſeries $Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. $= Ax + Bx^2 + \left(\frac{2B^2}{3A} - \frac{PB}{3} + A \times \frac{A^2+P^2}{6}\right)x^3 + \frac{2B^3 - 2PAB^2 + A^2 \times \overline{A^2+P^2} \times B}{6A^2}\Big)x^4 +$ &c., and the ſeries $1 + Px + Qx^2 + Rx^3 +$ &c. $= 1 + Px + \frac{A^2+P^2}{2}x^2 + \frac{2AB + P \times \overline{A^2+P^2}}{6}x^3 + \frac{4B^2 + \overline{A^2+P^2}^2}{24}x^4 +$ &c. Let $Kx^{n-2} + Lx^{n-1} + Mx^n$ be ſucceſſive terms of the ſeries $Ax + Bx^2 + Cx^3 +$ &c., and $K'x^{n-2} + L'x^{n-1} + M'x^n$ ſucceſſive terms of the ſeries $1 + Px + Qx^2 + Rx^3 +$ &c.; then will $A \times L + P \times L' = n \times M'$ and $B \times K + Q \times K' = n \cdot \frac{n-1}{2} \times M'$ expreſs the law of the ſerieſes.

Cor. Let $B=0$, then the ſeries $Ax+Bx^2+Cx^3+Dx^4=A\times$ $(x+\frac{P^2\times A^2}{2\cdot 3}x^3+\frac{(P^2+A^2)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5+\frac{(P^2+A^2)^3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\&c.)$, and the ſeries $1+Px+Qx^2+Rx^3+\&c.=1+Px+\frac{P^2+A^2}{1\cdot 2}x^2+P\times\frac{P^2+A^2}{1\cdot 2\cdot 3}x^3+\frac{(P^2+A^2)^2}{1\cdot 2\cdot 3\cdot 4}x^4+P\times\frac{(P^2+A^2)^2}{1\cdot 2\cdot 3\cdots 5}x^5+\frac{(P^2+A^2)^3}{1\cdot 2\cdot 3\cdot\cdot 6}x^6$ $+\&c.$; the co-efficient of the term x^n will be $(P^2+A^2)^{\frac{n}{2}}$ or $P\times(P^2+A^2)^{\frac{n-1}{2}}$, according as n is even or odd.

If in the equations before given for x be ſubſtituted $a-b$ inſtead of $a+b$, then in the other quantities for b ſubſtitute $-b$.

3. If in Caſe 2. the difference between the two quantities $(1+Pa+Qa^2+\&c.)\times(1+Pb+Qb^2+\&c.)$ and $(Aa+Ba^2+Ca^2+\&c.)\times(Ab+Bb^2+Cb^2+\&c.)$ is aſſumed $=1+P\times\overline{a+b}+Q\times\overline{a+b}^2+\&c.$, then in the ſerieſes before given for A, B, C, &c. write reſpectively $\sqrt{-1}A$, $\sqrt{-1}B$, $\sqrt{-1}C$, &c., and there will reſult the correſponding ſerieſes.

The ſame principles may be applied to many other caſes.

4. Equations of theſe formulæ may be uſeful, when the ſums of the ſerieſes correſpondent to a value (a) of x are given, and the ſums of the ſeries correſpondent to a value $(a+b)$ of x is required, b having a ſmall ratio to a: for inſtance, let the given ſeries be $x-\frac{x^3}{2\cdot 3}+\frac{x^5}{2\cdot 3\cdot 4\cdot 5}-\frac{x^7}{2\cdot 3\cdot\cdot 7}+\&c.$; the equation found in the firſt caſe is $a+b-\frac{(a+b)^3}{2\cdot 3}+\frac{(a+b)^5}{2\cdot 3\cdot 4\cdot 5}-\&c.=(a-\frac{a^3}{2\cdot 3}+\frac{a^5}{2\cdot 3\cdot 4\cdot 5}-\&c.)\times(1-\frac{b^2}{1\cdot 2}+\frac{b^4}{1\cdot 2\cdot 3\cdot 4}-\&c.)$ $+(1-\frac{a^2}{1\cdot 2}+\frac{a^4}{1\cdot 2\cdot 3\cdot 4}-\&c.)\times(b-\frac{b^3}{2\cdot 3}+\frac{b^5}{2\cdot 3\cdot 4\cdot 5}-\&c.)$;

but $a-\frac{a^3}{2\cdot3}+\frac{a^5}{2\cdot3\cdot4\cdot5}-$&c., and $1-\frac{a^2}{1\cdot2}+\frac{a^4}{2\cdot3\cdot4}-$&c. are the ſine ($s$) and coſine ($c$) of an arc a of a circle whoſe radius is 1; and, conſequently, if the ſine s and coſine c of an arc a be given, the ſine of an arc $(a+b)=s\times(1-\frac{b^2}{2}+\frac{b^4}{24}-$&c.$)+c(b-\frac{b^3}{2\cdot3}+\frac{b^5}{2\cdot3\cdot4\cdot5}-$&c.), which ſeries, if b be very ſmall in proportion to a, converges much faſter than the common ſeries for finding the ſine from the arc: it has been given from different principles in the Meditationes, and is alſo eaſily deducible from the ſeries for finding the ſine and coſine from the arc by the propoſitions uſually given in plane trigonometry: the coſine of the ſame arc $(a+b)=c\times(1-\frac{b^2}{1\cdot2}+\frac{b^4}{2\cdot3\cdot4}-$&c.) $-s\times(b-\frac{b^3}{1\cdot2\cdot3}+\frac{b^5}{1\cdot2\cdot\cdot5}-$&c.),

Ex. 2. Let the ſeries be $\overline{a+b}+\frac{\overline{a+b}^3}{2\cdot3}+\frac{\overline{a+b}^5}{2\cdot3\cdot4\cdot5}+$&c. $=(a+\frac{a^3}{2\cdot3}+\frac{a^5}{2\cdot3\cdot4\cdot5}+$&c.$\times(1+\frac{b^2}{1\cdot2}+\frac{b^4}{1\cdot2\cdot3\cdot4}+$&c.$)+(1+\frac{a^2}{1\cdot2}+\frac{a^4}{1\cdot2\cdot3\cdot4}+$&c.$)\times(b+\frac{b^3}{1\cdot2\cdot3}+\frac{b^5}{2\cdot3\cdot4\cdot5}+$&c.); but $a+\frac{a^3}{1\cdot2\cdot3}+$&c.$)=x$, and $1+\frac{a^2}{1\cdot2}+\frac{a^4}{1\cdot2\cdot3\cdot4}+$&c. $=\sqrt{1+x^2}$, if a be the hyperbolic log. of $x+\sqrt{1+x^2}$; therefore $a+b+\frac{\overline{a+b}^3}{2\cdot3}+\frac{\overline{a+b}^5}{2\cdot3\cdot4\cdot5}+$&c. $=x\times(1+\frac{b^2}{2}+\frac{b^4}{2\cdot3\cdot4}+$&c.$)+\sqrt{1+x^2}\times(b+\frac{b^3}{2\cdot3}+$&c.

Let $b+\frac{b^3}{2\cdot3}+\frac{b^4}{2\cdot3\cdot4\cdot5}+$&c. $=y$, and $(x+\sqrt{1+x^2}\times(y+\sqrt{1+y^2})=V$, then will $\overline{a+b}+\frac{\overline{a+b}^3}{2\cdot3}+\frac{\overline{a+b}^5}{2\cdot3\cdot4\cdot5}+$&c. $=\overline{\tfrac{1}{2}V-\frac{1}{V}}$.

5. Let

5. Let a quantity P be a function of x, or the fluent of a function of $x \times \dot{x}$, and the value X of it when $x=a$ be known, and the value of it when $x=a+b$ be required. Find a ſeries of which the firſt term is X, and which proceeds according to the dimenſions of b, if b be a very ſmall quantity, and in general at leaſt ſo ſmall that the ſeries from $x=a$ to $x=a+b$ neither becomes infinite or o.

In the ſame manner, if an algebraical or fluxional equation or equations, expreſſing the relations between x, y, z, v, &c. be given, find the correſpondent values of y, z, v, &c. to $x=a$, which let be Y, Z, V, &c.; then find ſerieſes for y, z, v, &c. of which the firſt terms let be Y, Z, V, &c. reſpectively, and which proceed according to the dimenſions of b, but ſubject to the ſame conditions as in the preceding caſe.

From fluxional equations may be deduced ſeries which expreſs the value of y, &c. in terms of x, and always diverge, or always converge, whatever may be its value, as appears from the Meditationes.